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Short Communication

Frequency analysis of axially loaded stepped beams by Green's function method

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Abstract

This paper presents a solution to a free vibration problem of stepped beams loaded by axial forces. The frequency equation is obtained by using properties of the Green's functions corresponding to uniform segments of the beam. The approach pertains to the vibration of beams consisting of an arbitrary number of uniform segments. The method can be used to obtain an approximate solution to vibration problems of beams with continuously varying cross-sections. Numerical examples are presented to demonstrate the usefulness of the method in the frequency analysis of stepped beams.

1. Introduction

The vibration problems of stepped beams are very important because of their practical significance in engineering applications. Stepped beams can be used to model the vibration of a robot arm, crane boom, tall building, etc. The vibration problems of axially loaded stepped Euler–Bernoulli beams are considered in Refs. [1–3]. Papers [1,2] deal with free vibrations of stepped beams partially stressed with follower forces. The force acts on one segment of a beam, consisting of two uniform segments. In order to solve the problem the authors of paper [1] used the Galerkin finite element formulation and the method proposed by Kikuchi. The author of paper [2] obtained the frequency equation by means of an exact approach. In Ref. [3], the free vibration of a beam with one step is also considered. The beam is loaded by a compressive or tensile axial force which changes stepwise at the step. The exact solution to the problem and the numerical results of the frequencies and critical forces are presented. The solution of free vibration problems of beams with up to three steps, varying in cross-section without an axial force, is presented in Ref. [4]. The frequency parameters are tabulated for beams with various types of end supports.

The solution to vibration problems of stepped beams can be obtain by using Green's function method (GFM). The advantage of using this method is best seen in the case of a stepped beam consisting of a high number of uniform segments. The GFM was applied in paper [5] where the vibration problems of stepped beams and rectangular plates are discussed, but no consideration is given to the vibration of axially loaded

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beams. In Ref. [6], this method was used in the free vibration analysis of axially loaded beams with discrete elements attached, but the analysis only concerns the vibration of uniform beams.

The subject of our paper is the problem of the free vibration of an axially loaded stepped beam. The formulation of and solution to the problem concern a stepped Euler–Bernoulli beam consisting of an arbitrary number of segments with constant cross-sections. Additionally, we also considered the vibration of a cantilever stepped beam which includes a tip mass and/or an elastic support at the free end. The analytical solution to the problem is obtained by the application of GFM. The solution is then used in the numerical calculation of the eigenfrequencies of the stepped beams. Exemplary numerical results are given to shown the application of the presented approach to obtain the approximate solution to vibration problems of beams with continuously varying cross-sections.

2. Theory

Consider a stepped beam which consists of n segments with constant cross-sections (Fig. 1). The governing equations for the transverse displacements w_i of the segments are

$$\mathscr{L}_1[w_1(x,t)] = s_1(t)\,\delta(x-L_1) - m_1(t)\delta'(x-L_1) \quad \text{for } x \in [0, L_1], \tag{1}$$

$$\mathscr{L}_i[w_i(x,t)] = -s_{i-1}(t)\,\delta(x-L_{i-1}) + m_{i-1}(t)\,\delta'(x-L_{i-1}) + s_i(t)\,\delta(x-L_i) - m_i(t)\delta'(x-L_i)$$
$$\text{for } x \in [L_{i-1}, L_i], i = 2, \ldots, n-1, \tag{2}$$

$$\mathscr{L}_n[w_n(x,t)] = -s_{n-1}(t)\,\delta(x-L_{n-1}) + m_{n-1}(t)\,\delta'(x-L_{n-1}) + f(x,t) \text{ for } x \in [L_{n-1}, L], \tag{3}$$

where $\mathscr{L}_i$ is a differential operator:

$$\mathscr{L}_i = (EI)_i \frac{\partial^4}{\partial x^4} + p_i \frac{\partial^2}{\partial x^2} + (\rho A)_i \frac{\partial^2}{\partial t^2} \quad \text{for } i = 1, \ldots, n,$$

$\delta(\cdot)$ is Dirac's delta function, $\delta'(\cdot)$ is the doublet function, p_i is the axial load acting on the ith segment of the beam, $(\rho A)_i$ and $(EI)_i$ are the mass per unit length and the flexural rigidity of the ith segment, respectively. Function $f(x,t)$ occurring in Eq. (3) denotes a force per unit length of the beam, while $s_i(t)$ and $m_i(t)$, respectively, represent the shear force and bending moment acting on the right end of the ith segment. A schematic diagram of the ith segment of the considered beam is shown in Fig. 2.

Functions w_1 and w_n, which describe the transverse displacements of the edge segments, satisfy boundary conditions which depend on the constraints of the beam ends. The conditions may be written symbolically in the form

$$\mathscr{B}_0[w_1]\big|_{x=0} = 0, \quad \mathscr{B}_1[w_n]\big|_{x=L} = 0, \tag{4}$$

where $\mathscr{B}_0$ and $\mathscr{B}_1$ are two-dimensional "vectors", the components of which are linear, spatial differential operators. Moreover, the continuity of displacements and slopes indicate the following conditions:

$$w_i(L_i,t) = w_{i+1}(L_i,t), \quad w_{i,x}(L_i,t) = w_{i+1,x}(L_i,t), \quad i = 1, \ldots, n-1. \tag{5}$$

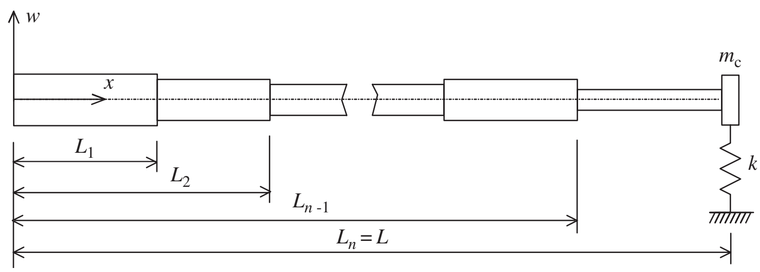

Fig. 1. A sketch of the considered system.

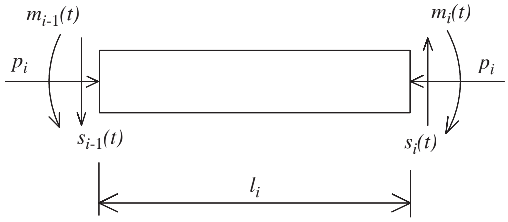

Fig. 2. A sketch of the ith segment of the stepped beam.

For the free vibration of the beam, we assume that

$$w_i(x,t)=\overline{W}_i(x)\mathrm{e}^{\mathrm{j}\omega t},\quad s_i(t)=\overline{S}_i\mathrm{e}^{\mathrm{j}\omega t},\quad m_i(t)=\overline{M}_i\mathrm{e}^{\mathrm{j}\omega t}f(x,t)=\overline{F}(x)\mathrm{e}^{\mathrm{j}\omega t},\tag{6}$$

where $\mathrm{j}=\sqrt{-1}$ and ω is the natural frequency of the beam. Substituting (6) into Eqs. (1)–(5), obtains

$$\tilde{\mathscr{L}}_1[W_1(\xi_1)]=S_1\delta(\xi_1-1)-M_1\delta'(\xi_1-1),\tag{7}$$

$$\tilde{\mathscr{L}}_i[W_i(\xi_i)]=-\sigma_{i-1}S_{i-1}\delta(\xi_i)+\mu_{i-1}M_{i-1}\delta'(\xi_i)+S_i\delta(\xi_i-1)-M_i\delta'(\xi_i-1),\quad i=2,\dots,n-1,\tag{8}$$

$$\tilde{\mathscr{L}}_n[W_n(\xi_n)]=-\sigma_{n-1}S_{n-1}\delta(\xi_n)+\mu_{n-1}M_{n-1}\delta'(\xi_n)+F(\xi_n),\tag{9}$$

where

$$\tilde{\mathscr{L}}_i=\frac{\mathrm{d}^4}{\mathrm{d}\xi_i^4}+P_i\frac{\mathrm{d}^2}{\mathrm{d}\xi_i^2}-\Omega_i^4$$

for $\xi_i\in[0,1]$, $\xi_i=(x-L_{i-1})/l_i$, $l_i=L_i-L_{i-1}$ is the length of the ith beam segment, $W_i=\overline{W}_i/l_i$, $P_i=p_il_i^2/(EI)_i$, $S_i=\overline{S}_il_i^2/(EI)_i$, $M_i=\overline{M}_il_i^3/(EI)_i$, $\Omega_i^4=(\rho A)_i\omega^2l_i^4/(EI)_i$, $\lambda_i=l_{i+1}/l_i$, $\mu_i=\lambda_i^3(EI)_i/(EI)_{i+1}$, $\sigma_i=\lambda_i^2(EI)_i/(EI)_{i+1}$. The last term in Eq. (9) represents an added discrete element (a concentrated mass and/or a spring) attached to the beam at its free end. Therefore, this function assumes the form [6]

$$F(\xi_n)=(M_c\Omega_n^4-K)W_n(\xi_n)\delta(\xi_n-1),\tag{10}$$

where $K=kl_n^3/(EI)_n$, $M_c=m_c/((\rho A)_nl_n)$, k is the stiffness coefficient of the translational spring, m_c is the concentrated mass attached at the beam end. From Eqs. (4)–(6) it follows that functions W_i satisfy the conditions below:

$$\tilde{\mathscr{B}}_0[W_1]\big|_{\xi_1=0}=0,\quad \tilde{\mathscr{B}}_1[W_n]\big|_{\xi_n=1}=0,\tag{11}$$

$$W_i(1)=\lambda_iW_{i+1}(0),\quad W_{i,\xi_i}(1)=W_{i+1,\xi_{i+1}}(0),\quad i=1,\dots,n-1.\tag{12}$$

3. Solution to the problem

The solution to the free vibration problem is obtained by using GFM. To this end we need Green's functions G_i which satisfy the differential equations

$$\tilde{\mathscr{L}}_i[G_i]=\delta(\xi_i-\eta),\quad i=1,\dots,n\tag{13}$$

and the following boundary conditions:

$$\tilde{\mathscr{B}}_0[G_1]\big|_{\xi_1=0}=0,\quad G_{1,\xi_1\xi_1}\big|_{\xi_1=1}=(G_{1,\xi_1\xi_1\xi_1}+P_1G_{1,\xi_1})\big|_{\xi_1=1}=0,\tag{14}$$

$$G_{i,\xi_i\xi_i}\big|_{\xi_i=0}=(G_{i,\xi_i\xi_i\xi_i}+P_iG_{i,\xi_i})\big|_{\xi_i=0}=0,\quad G_{i,\xi_i\xi_i}\big|_{\xi_i=1}=(G_{i,\xi_i\xi_i\xi_i}+P_iG_{i,\xi_i})\big|_{\xi_i=1}=0\quad \text{for } i=2,3,\dots,n-1,\tag{15}$$

$$G_{n,\xi_n\xi_n}\big|_{\xi_n=0} = (G_{n,\xi_n\xi_n\xi_n} + P_n G_{n,\xi_n})\big|_{\xi_n=0} = 0, \quad \tilde{\mathscr{B}}_1[G_n]\big|_{\xi_n=1} = 0. \tag{16}$$

To present the solution to problem (13)–(16), four linearly independent solutions to a homogeneous differential equation associated with the fourth-order Eq. (13) are needed. It is simple to prove that the following four functions satisfy the homogeneous equation associated with Eq. (13):

$$\phi_i^{(0)}(\xi_i) = \cosh\beta_i\xi_i - \cos\alpha_i\xi_i, \quad \phi_i^{(1)}(\xi_i) = \beta_i \sinh\beta_i\xi_i + \alpha_i \sin\alpha_i\xi_i,$$
$$\phi_i^{(2)}(\xi_i) = \beta_i^2 \cosh\beta_i\xi_i + \alpha_i^2 \cos\alpha_i\xi_i, \quad \phi_i^{(3)}(\xi_i) = \beta_i^3 \sinh\beta_i\xi_i - \alpha_i^3 \sin\alpha_i\xi_i,$$

where $\alpha_i = \sqrt{(1/2)(P_i + \sqrt{P_i^2 + 4\Omega_i^4})}$, $\beta_i = \sqrt{(1/2)(-P_i + \sqrt{P_i^2 + 4\Omega_i^4})}$. In order to show that these functions are linearly independent, the Wronskian should be determined. After calculation, the Wronskian is obtained in the form: $-\Omega_i^4(P_i^2 + 4\Omega_i^4)^2$. Because the Wronskian is not equal to zero for $\Omega_i > 0$, then functions $\phi_i^{(j)}(\xi_i)$ $(j = 0, \ldots, 3)$ constitute a fundamental set of solutions to the homogenous equation associated with (13). Therefore, the general solution to Eq. (13) may be written as

$$G_i(\xi_i, \eta_i; \Omega_i) = \sum_{j=1}^{4} C_{ij}(\eta_i)\phi_i^{(j-1)}(\xi_i) + G_{pi}(\xi_i, \eta_i), \quad i = 1, \ldots, n, \tag{17}$$

where $G_{pi}(\xi_i, \eta_i)$ denotes a particular solution to Eq. (13). This solution can be presented in the form [6]

$$G_{pi}(\xi_i, \eta_i) = \bar{\phi}_i^{(1)}(\xi_i - \eta_i)H(\xi_i - \eta_i),$$

where H denotes a Heaviside function and

$$\bar{\phi}_i^{(1)}(\xi_i) = \frac{\sinh\beta_i\xi_i}{\beta_i} - \frac{\sinh\alpha_i\xi_i}{\alpha_i}.$$

The four constants C_{ij} $(j = 1, \ldots, 4)$ occurring in the general solution (17) are determined on the basis of the boundary conditions. Two of the four constants can be eliminated by using the boundary conditions which are satisfied at $\xi_i = 0$. For example, after eliminating the two constants, the Green's function G_1 corresponding to a beam with the left end clamped ($G_1|_{\xi_1=0} = G_{1,\xi_1}|_{\xi_1=0} = 0$), may be expressed by the formula:

$$G_1(\xi_1, \eta_1; \Omega_1) = C_{11}(\eta_1)\phi_1(\xi_1) + C_{12}(\eta_1)\phi_1^{(1)}(\xi_1) + \bar{\phi}_1^{(1)}(\xi_1 - \eta_1)H(\xi_1 - \eta_1). \tag{18}$$

Similarly, using Eq. (17) and conditions (15a) or (16a), the Green's functions G_i corresponding to a beam with the left end free may be written as follows:

$$G_i(\xi_i, \eta_i; \Omega_i) = C_{i1}(\eta_i)\phi_i^{(1)}(\xi_i) + C_{i2}(\eta_i)\phi_i^{(2)}(\xi_i) + \bar{\phi}_i^{(1)}(\xi_i - \eta_i)H(\xi_i - \eta_i), \quad i = 1, 2, \ldots, n, \tag{19}$$

where

$$\bar{\phi}_i^{(2)}(\xi_i) = \frac{\cosh\beta_i\xi_i}{\beta_i^2} + \frac{\cos\alpha_i\xi_i}{\alpha_i^2}.$$

Functions $C_{i1}(\eta_i)$ and $C_{i2}(\eta_i)$ in Eqs. (18) and (19) are determined by using boundary conditions for $\xi_i = 1$. For example, when the right end of the beam is free, the functions (using conditions (14b) and (15b)) can be written in the form

$$C_{11}(\eta_1) = \frac{1}{D_1}\left\{\phi_1^{(1)}(1)\bar{\phi}_1^{(2)}(1 - \eta_1) - \phi_1^{(1)}(1 - \eta_1)\bar{\phi}_1^{(2)}(1)\right\},$$

$$C_{12}(\eta_1) = -\frac{1}{D_1}\left\{\phi_1^{(2)}(1)\bar{\phi}_1^{(2)}(1 - \eta_1) - \phi_1^{(1)}(1 - \eta_1)\bar{\phi}_1^{(1)}(1)\right\},$$

$$C_{i1}(\eta_i) = \frac{1}{D_i}\left\{\phi_i^{(0)}(1)\bar{\phi}_i^{(2)}(1 - \eta_i) - \phi_i^{(1)}(1 - \eta_i)\bar{\phi}_i^{(3)}(1)\right\},$$

$$C_{i2}(\eta_i) = \frac{1}{D_i}\left\{\phi_i^{(0)}(1)\phi_i^{(1)}(1-\eta_i) - \bar{\phi}_i^{(2)}(1-\eta_i)\phi_i^{(3)}(1)\right\},$$

where $D_1 = \phi_1^{(2)}(1)\bar{\phi}_1^{(2)}(1) - \phi_1^{(1)}(1)\bar{\phi}_1^{(1)}(1)$, for the cantilever beam, $D_i = \phi_i^{(3)}(1)\bar{\phi}_i^{(3)}(1) - [\phi_i^{(0)}(1)]^2$, for the free beam.

The Green's functions corresponding to uniform beams with other boundary conditions can be similarly determined.

The solutions to Eqs. (7)–(9) which satisfy boundary conditions (11) may be expressed by the Green's functions as follows:

$$W_1(\xi_1) = S_1 G_1(\xi_1, 1; \Omega_1) + M_1 G_{1,\eta_1}(\xi_1, 1; \Omega_1), \tag{20}$$

$$\begin{aligned} W_i(\xi_i) = & -\sigma_{i-1} S_{i-1} G_i(\xi_i, 0; \Omega_i) - \mu_{i-1} M_{i-1} G_{i,\eta_i}(\xi_i, 0; \Omega_i) \\ & + S_i G_i(\xi_i, 1; \Omega_i) + M_i G_{i,\eta_i}(\xi_i, 1; \Omega_i) \quad \text{for } i = 2, \ldots, n-1, \end{aligned} \tag{21}$$

$$\begin{aligned} W_n(\xi_n) = & -\sigma_{n-1} S_{n-1} G_n(\xi_n, 0; \Omega_n) - \mu_{n-1} M_{n-1} G_{n,\eta_n}(\xi_n, 0; \Omega_n) \\ & + \left(M_c \Omega_n^4 - K\right) W_n(1) G_n(\xi_n, 1; \Omega_n). \end{aligned} \tag{22}$$

Substituting functions W_i into continuity conditions (12), we obtain a set of equations with the unknowns: $S_1, M_1, S_2, M_2, \ldots, S_{n-1}, M_{n-1}, W_n(1)$. The equations are completed by adding an equation which is obtained by assuming $\xi_n = 1$ in Eq. (22). This set of the equations may be written in the following matrix form:

$$\mathbf{A}(\omega)\mathbf{x} = \mathbf{0}, \tag{23}$$

where $\mathbf{x} = [S_1\ M_1\ \ldots\ S_{n-1}\ M_{n-1}\ W_n(1)]^{\mathrm{T}}$ and $\mathbf{A}(\omega) = [a_{ij}]_{1 \leqslant i,j \leqslant 2n-1}$. The non-zero elements a_{ij} of matrix $\mathbf{A}$ are given in Appendix A.

A non-trivial solution to Eq. (23) exists when the determinant of matrix $\mathbf{A}$ is set equal to zero, yielding the frequency equation of the stepped beam:

$$\det \mathbf{A}(\omega) = 0. \tag{24}$$

If the Green's function G_1 occurring in this equation corresponds to the clamped–free beam and functions G_i $(i = 2, \ldots, n)$ correspond to the free–free beams, then the equation corresponds to a stepped cantilever beam. After assuming function G_1 corresponding to the free–free beam, the frequency equation for the beam with left end free can be obtained. If parameter K, which characterize the stiffness of the elastic support on the right end of the beam, tends to infinity in Eq. (24), then the frequency equation corresponding to the beam with a pinned right end is obtained.

Frequency Eq. (24) with respect to the circular frequency ω, is then solved numerically by using an approximate method.

4. Numerical examples

The presented procedure was proved by comparing the numerical results obtained here with the results presented in the literature. The eigenfrequencies of the clamped–free (c–f) and free–pined (f–p) beams with one, two or three steps were calculated with the use of Eq. (24) and are given in Table 1. The calculations were performed for beams without any support or attached mass. The uniform segments of the beam with one step are loaded by a compressive ($P_1 = 10$, $P_2 = 10$) or tensile ($P_2 = -5$) force, the beams with three or four segments are considered without any axial load. The frequencies obtained by Naguleswaran in Ref. [3] (for one-step beam, $n = 2$) and [4] (for two- and three-step beams, $n = 3$ or 4) are written in parentheses.

Next, we investigated the effect of the axial force acting on one segment of the one-step cantilever on the natural frequency of the beam. A beam with or without a concentrated mass or elastic support at the free end was considered. The results were obtained for a stepped beam consisting of two uniform segments of exactly the same length but differing in the widths of the rectangular cross-sections. The frequency curves are shown in Fig. 3. The continuous lines refer to a beam with segments for which $\gamma = A_2/A_1 = I_2/I_1 = 1.0$, the dotted lines concern a beam with $\gamma = 0.75$ and the dashed lines refer to a beam with $\gamma = 0.5$. The letters labelling the

Table 1
Five dimensionless eigenfrequencies, $\Omega_{1,j} = \sqrt[4]{(\rho A)_1 L^4 \omega_j^2/(EI)_1}$, of clamped–free (c–f) and free–pinned (f–p) beams with $n = 2$ or 3 or 4 uniform segments (in parentheses are given the results presented in Refs. [3] or [4])

BC	N	P_1	P_2	$\Omega_{1,1}$	$\Omega_{1,2}$	$\Omega_{1,3}$	$\Omega_{1,4}$	$\Omega_{1,5}$
c–f	2	10	10	2.96814 (2.9681)	5.40792 (5.4079)	7.98020 (7.9802)	10.74571	13.45520
c–f	2	10	−5	3.88793 (3.8879)	7.00962 (7.0096)	10.14408 (10.1441)	13.01341	16.01630
c–f	3	0	0	2.02880 (2.02880)	3.59685 (3.59685)	5.29414 (5.29414)	7.81612	8.97501
c–f	4	0	0	2.51010 (2.51010)	4.44542 (4.44542)	5.81961 (5.81961)	8.57074	11.39880
f–p	2	10	10	2.33785 (2.3378)	4.66459 (4.6646)	7.18013 (7.1801)	10.03578	12.75424
f–p	2	10	−5	3.90380 (3.9038)	6.57734 (6.5773)	9.57250 (9.5725)	12.42295	15.31436
f–p	3	0	0	1.54446 (1.54446)	4.43732 (4.43732)	7.35818 (7.35818)	8.31478	11.33268
f–p	4	0	0	2.13937 (2.13937)	5.22801 (5.22801)	8.21372 (8.21372)	10.11844	12.43464

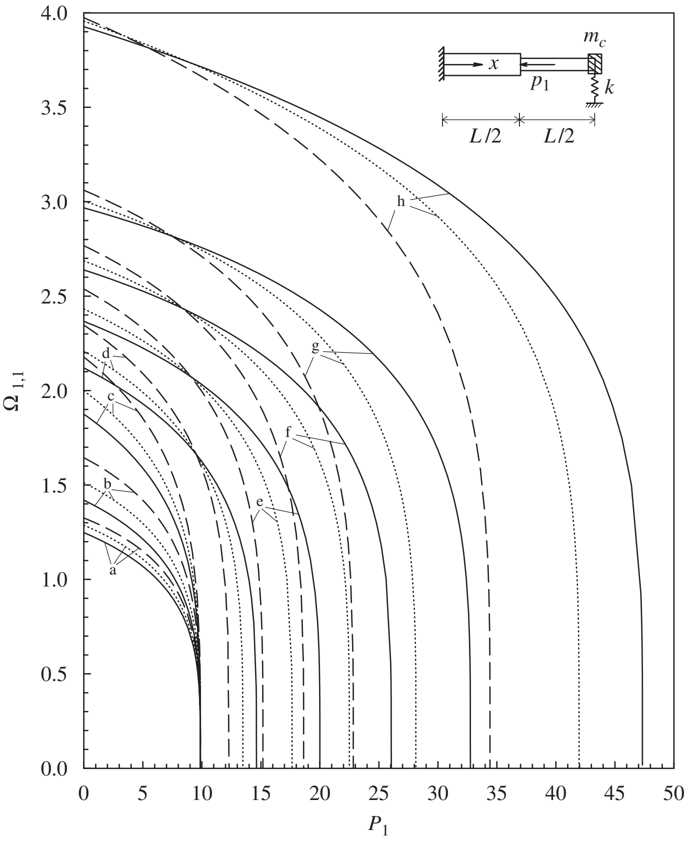

Fig. 3. Eigenfrequency parameter values $\Omega_{1,1} = \sqrt[4]{(\rho A)_1 L^4 \omega_1^2/(EI)_1}$ as a function of the axial force $P_1 = p_1 L^2/(EI)_1$ for one-step cantilever beam with or without a concentrated mass or elastic support at the free end (description of the curves a–h is given in Table 2); —— $\gamma = 1.0$, · · · · · · · · · $\gamma = 0.75$, – – – – $\gamma = 0.5$.

curves relate to the concentrated mass or elastic support attached at the free end of the beam. The non-dimensional values of the concentrated mass, $\bar{M}_c = m_c/m_b$ where $m_b = (\rho A)_1 l_1 + (\rho A)_2 l_2$, and the stiffness coefficient of the elastic support, $\bar{K} = kL^3/(EI)_2$, which were used in the numerical calculations, are presented in Table 2. The results show that both the tip mass as well as the change in the ratio $\gamma = A_2/A_1 = I_2/I_1$ influence the frequency of the stepped beam but do not change the critical load. In the case of a beam with an

Table 2
The non-dimensional values of the stiffness of the elastic support, $\bar{K} = kL^3/(EI)_2$, and the concentrated mass, $\bar{M}_c = m_c/m_b$, used in numerical calculations of the frequency curves shown in Fig. 3

	Description of the curves in Fig. 3							
	a	b	c	d	e	f	g	h
$\bar{K}$	0	0	0	2	5	10	20	∞
$\bar{M}_c$	$1/(1+\gamma)$	1	0	0	0	0	0	0

Table 3
Five dimensionless eigenfrequencies $\Omega_{1,j} = \sqrt[4]{(\rho A)_1 L^4 \omega_j^2/(EI)_1}$ of the cantilever beam with n uniform segments of constant height and width changing at the steps (the results presented in Ref. [8] are given in parentheses)

$A/A_0 = I/I_0$	n	$\Omega_{1,1}$	$\Omega_{1,2}$	$\Omega_{1,3}$	$\Omega_{1,4}$	$\Omega_{1,5}$
$1-x/2$	5	2.05727	4.81849	7.90833	11.01534	14.13583
	10	2.07067	4.84027	7.93754	11.04969	14.17346
	20	2.07597	4.84754	7.94681	11.06122	14.18714
	40	2.07824 (2.07730)	4.85013 (4.89666)	7.94970 (7.94979)	11.06465 (11.06524)	14.19118
$1+x+x^2$	5	1.58284	4.48133	7.77289	10.96067	14.11320
	10	1.57436	4.46148	7.74273	10.92592	14.09137
	20	1.57250	4.45642	7.73422	10.91452	14.07746
	40	1.57217 (1.57187)	4.45518 (4.45474)	7.73196 (7.73137)	10.91144 (10.91059)	14.07362

elastic support at the free end it can be seen that an increase in ratio γ causes an increase in the critical load. If the stiffness coefficient K tends to infinity, then the results refer to clamped–pinned beams (the curves labelled "*h*" in Fig. 3).

A beam with a continuously varying cross-section can be approximated by a stepped beam with any number of uniform segments. This approximation of Timoshenko beams was used in paper [7]. The vibration frequencies of Euler–Bernoulli beams with varying cross-sections are presented in tabular form, for example, by Abrate in Ref. [8]. A comparison of the frequencies obtained in Ref. [8] and those obtained by using frequency Eq. (24) for various numbers of uniform segments is presented in Table 3 (the numbers in parentheses are the square roots of the non-dimensional frequencies given in Ref. [8]). The calculations were performed for a cantilever beam of rectangular cross-section without an axial force. Two cases of the beam are considered: a beam with constant height and linearly varying width ($A/A_0 = I/I_0 = 1 - x/2$) and a beam with constant height and quadratically varying width ($A/A_0 = I/I_0 = 1 + x + x^2$), where A_0, I_0 are the cross-sectional area and the moment of inertia at $x = 0$, respectively. The width of the ith segment of the stepped beam is equal to the width of the non-uniform beam in the middle of the segment, and the heights of the beams are the same. The results presented in the table show that the eigenfrequencies become increasingly accurate as the number of uniform segments in the approximated stepped beam increase.

5. Conclusions

The paper presents the application of the GFM in the frequency analysis of axially loaded stepped beams. The exact transcendental frequency equation was solved by using the numerical method. Stepped beams with any number of uniform segments may be used as an approximation of non-uniform beams with continuously varying cross-section areas. The example shows that the accuracy of the numerically obtained eigenfrequencies improves as the step number of the stepped beams increases. Although the examples presented in this paper concern clamped–free and free–pinned stepped beams, the approach may be used in the vibration analysis of stepped beams with other boundary conditions.

Acknowledgments

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Appendix A

The non-zero elements a_{ij} of the matrix **A** which occurs in Eq. (24) are as follows (n is number of uniform segments of the stepped beam):

for $i = 1, 2, \ldots, n-1$: $a_{2i-1,2i-1} = G_i(1, 1; \Omega_i) + \lambda_i \sigma_i G_{i+1}(0, 0; \Omega_{i+1})$,
$a_{2i-1,2i} = G_{i,\xi_i}(1, 1; \Omega_i) + \lambda_i \sigma_i G_{i+1,\xi_{i+1}}(0, 0; \Omega_{i+1})$,
$a_{2i,2i-1} = G_{i,\xi_i}(1, 1; \Omega_i) + \lambda_i \sigma_i G_{i+1,\xi_{i+1}}(0, 0; \Omega_{i+1})$,
$a_{2i,2i} = G_{i,\eta_i \xi_i}(1, 1; \Omega_i) + \sigma_i G_{i+1,\eta_{i+1}\xi_{i+1}}(0, 0; \Omega_{i+1})$;

for $i = 1, 2, \ldots, n-2$:
$a_{2i-1,2i+1} = -\lambda_i G_{i+1}(0, 1; \Omega_{i+1})$, $a_{2i-1,2i+2} = -\lambda_i G_{i+1,\xi_{i+1}}(0, 1; \Omega_{i+1})$,
$a_{2i,2i+1} = -G_{i+1,\xi_{i+1}}(0, 1; \Omega_{i+1})$, $a_{2i,2i+2} = -G_{i+1,\eta_{i+1}\xi_{i+1}}(0, 1; \Omega_{i+1})$;

for $i = 2, 3, \ldots, n-1$: $a_{2i-1,2i-3} = -\sigma_{i-1} G_i(1, 0; \Omega_i)$, $a_{2i-1,2i-2} = -\mu_{i-1} G_{i,\xi_i}(1, 0; \Omega_i)$,
$a_{2i,2i-3} = -\sigma_{i-1} G_{i,\xi_i}(1, 0; \Omega_i)$, $a_{2i,2i-2} = -\mu_{i-1} G_{i,\eta_i \xi_i}(1, 0; \Omega_i)$

and
$a_{2n-3,2n-1} = -(K - M_c \Omega_n^4)\lambda_{n-1} G_n(0, 1; \Omega_n)$, $a_{2n-2,2n-1} = -(K - M_c \Omega_n^4) G_{n,\xi_n}(0, 1; \Omega_n)$,
$a_{2n-1,2n-3} = \sigma_{n-1} G_n(1, 0; \Omega_n)$, $a_{2n-1,2n-2} = \mu_{n-1} G_{n,\xi_n}(1, 0; \Omega_n)$, $a_{2n-1,2n-1} = 1 + (K - M_c \Omega_n^4) G_n(1, 1; \Omega_n)$.

References

[1] K. Kanaka Raju, G. Venkateswara Rao, Free vibrations of a free–free partially stressed stepped beam with follower forces, *Journal of Sound and Vibration* 178 (1994) 429–432.

[2] M.A. De Rosa, Free vibration of stepped beams with flexible ends, in the presence of follower forces at the step, *Journal of Sound and Vibration* 194 (1996) 631–635.

[3] S. Naguleswaran, Transverse vibration and stability of an Euler–Bernoulli beam with step change in cross-section and in axial force, *Journal of Sound and Vibration* 270 (2004) 1045–1055.

[4] S. Naguleswaran, Vibration of an Euler–Bernoulli beam on elastic end supports and with up to three step changes in cross-section, *International Journal of Mechanical Sciences* 44 (2002) 2541–2555.

[5] J. Lee, L.A. Bergman, The vibration of stepped beams and rectangular plates by an elemental dynamic flexibility method, *Journal of Sound and Vibration* 171 (1994) 617–640.

[6] S. Kukla, Free vibrations of axially loaded beams with concentrated masses and intermediate elastic supports, *Journal of Sound and Vibration* 172 (1994) 449–458.

[7] X. Tong, B. Tabarrok, K.Y. Yeh, Vibration analysis of Timoshenko beams with non-homogenity and varying cross-section, *Journal of Sound and Vibration* 186 (1995) 821–835.

[8] S. Abrate, Vibration of non-uniform rods and beams, *Journal of Sound and Vibration* 185 (1995) 703–716.